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A Systematic Review of Length-Biased Distributions: Theory, Developments and Applications

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Abstract

Length-biased distributions arise when the probability of including an observational unit in a sample is proportional to its length. They form an important special case of weighted distributions and have substantial applications in survival analysis, renewal processes, reliability studies, biomedical investigations, ecology, actuarial science and related areas. The present review provides a concise but systematic account of the theory and development of length-biased distributions, with emphasis on their mathematical formulation, historical development, statistical properties, inferential methods and applications. Special attention is given to models used in lifetime data analysis, including the Weibull length-biased distribution, Weibull length-biased exponential distribution, length-biased beta-Pareto distribution, length-biased weighted Lomax distribution, length-biased Sushila distribution, length-biased Aradhana distribution and length-biased transmuted models. The review also discusses nonparametric estimation, goodness-of-fit procedures, maximum likelihood and Bayesian estimation, and the comparative performance of length-biased models in lifetime data analysis. The study also classifies the existing literature and identifies potential areas for future research.

1. Introduction

The analysis of lifetime and reliability data frequently requires models capable of accommodating skewness, heterogeneity and non-random selection mechanisms. In many practical investigations, sampled observations do not arise from ordinary random sampling. Instead, units with larger length, duration, or size are more likely to be observed. In such situations, inference based on the original parent distribution may be misleading, and an appropriate weighted model becomes necessary.

Length-biased distributions constitute one of the most important subclasses of weighted distributions. Their origin may be traced to Fisher's work on ascertainment bias and Rao's later formal development of weighted distributions. The concept of length-biased sampling was further emphasized by Cox in the study of fiber lengths and subsequently found broad relevance in human populations, ecology, biomedicine, renewal theory and reliability analysis. In practical terms, if longer-lived units are more likely to be observed than shorter-lived units, the resulting sample no longer follows the original lifetime distribution rather, it follows its length-biased version.

Over the past two decades, substantial work has been devoted to constructing new length-biased models from classical baseline distributions. This article presents the theoretical foundation of length-biased distributions, traces their historical development, surveys key model families, summarizes their major statistical properties, reviews inferential tools and identifies directions for future research.

2. The Length-Biased Family of Distributions

The length-biased family of distributions constitutes an important subclass of weighted distributions, arising when the probability of selecting an observation is proportional to its magnitude or duration. This family is generated by modifying a baseline distribution through a weight function, typically proportional to the variable itself.

Let observation x has probability density function (pdf) $f(x)$ for a baseline distribution. Then, this family is generated by modifying a baseline distribution through a weight function, typically proportional to the variable itself, leading to the general form as

$$g(x) = \frac{xf(x)}{E(x)}, \qquad x > 0$$

A length-biased distribution reduces to the original (baseline) distribution when the weighting mechanism becomes constant**,** i.e., when there is no bias in sampling**.** The growing literature on length-biased families highlights their theoretical significance as well as practical effectiveness in handling biased sampling mechanisms.

3. Historical Development

The concept of length-biased distributions originates from the theory of weighted distributions, initially introduced by Fisher (1934) to address ascertainment bias and later formalized by Rao (1965) for modelling data under unequal selection probabilities. The concept of length biased sampling was first introduced by Cox (1969) and Zelen (1974). More generally, when the sampling mechanism, while Patil and Rao (1978) expanded its applications to fields such as ecology and human population studies. Subsequent research developed formal relationships between baseline and length-biased distributions and examined their effects on reliability measures and statistical inference.
In recent years, attention has shifted toward the construction of flexible length-biased models derived from classical distributions such as exponential, Weibull, Lomax, Pareto and beta families. These developments aim to improve goodness-of-fit, accommodate diverse hazard rate behaviors and enhance modelling of lifetime data.

4. Review of Major Length-Biased Models

A large portion of the available literature concerns the derivation of new length-biased forms of existing lifetime distributions. Some of these models are :

4.1 Weibull Length-Biased Distribution

The Weibull distribution is one of the most important models in reliability theory because of its ability to represent increasing and decreasing failure rates. Its length-biased version has been studied in detail with emphasis on shape, moments, reliability and estimation. The Weibull length-biased distribution is unimodal, and its hazard rate is reported to be upside-down bathtub shaped when the shape parameter is less than one and increasing when the shape parameter is at least one. Both classical and Bayesian estimation procedures have been discussed for this model, demonstrating its theoretical and practical importance.

4.2 Weibull Length-Biased Exponential Distribution

A further extension is the Weibull length-biased exponential distribution, proposed as a three-parameter model. Its appeal lies in combining the flexibility of Weibull-generated families with the practical relevance of length-biased sampling. The model has been studied through density expansions, moments, reliability functions, asymptotic behavior and maximum likelihood estimation. Simulation studies and applications to lifetime datasets indicate that this model can provide improved flexibility over simpler competitors.

4.3 Length-Biased Beta-Pareto Distribution

The beta-Pareto family is useful for heavy-tailed and skewed lifetime data. Its length-biased version broadens the modelling scope further and contains several submodels, including the length-biased Pareto and certain exponential-type cases. Studies on this model examined hazard behavior, Shannon and Rényi entropy measures and maximum likelihood estimation. Applications to insurance-related data suggest that the model is capable of fitting heavy-tailed observations effectively.

4.4 Length-Biased Weighted Lomax Distribution

The Lomax distribution has gained importance in business-failure analysis, actuarial science and heavy-tailed lifetime modelling. The length-biased weighted Lomax distribution was developed to address situations in which larger observations are overrepresented. Research on this model has focused on structural properties, estimation by maximum likelihood and application to real datasets. It has been shown to provide a useful alternative in reliability and lifetime modelling, especially for data with right-skewness and long tails.

4.5 Length-Biased Sushila Distribution

The Sushila distribution, originally introduced as a mixture-type lifetime model, has also been transformed into a length-biased version. The resulting distribution has two parameters and has been studied through reliability measures, entropies, order statistics, likelihood ratio testing and maximum likelihood estimation. Comparative empirical results indicate that the length-biased Sushila model can fit certain lifetime datasets better than the original Sushila distribution, judging by criteria such as AIC, BIC and log-likelihood.

4.6 Length-Biased Aradhana Distribution

The Aradhana distribution is a one-parameter lifetime model designed for engineering and biomedical applications. Its length-biased version has been proposed to improve flexibility and empirical performance. Reported applications show that the length-biased Aradhana distribution may outperform both the parent Aradhana and the exponential distribution on selected datasets. This reinforces a common theme in the literature: length-biasing often introduces a practically useful extra level of realism for data generated under unequal selection probabilities

4.7 Weighted Log-Uniform Distribution

Another recent addition to the length-biased family is the weighted log-uniform (WLU) distribution, which extends the classical log-uniform model by incorporating a weight function to account for length-biased sampling. For the datasets, naturally constrained between two values, a flexible model, namely weighted log uniform distribution is proposed. Simulation study shows

the increased accuracy of the estimators with respect to the sample size. The proposed distribution belongs to the IFR family of distributions, and has significant relevance in the reliability engineering.

4.8 Length-Biased Transmuted and Other Recent Models

Recent work has also considered length-biased versions of transformed or generated families. One example is the length-biased transmuted Mukherjee–Islam distribution, where the weighted-transformation approach is combined with transmutation. The associated study derived an extensive range of properties, including survival and hazard functions, cumulative hazard, mean residual life, moments, generating functions, order statistics, entropy and inequality curves.

5. Statistical Properties

A major objective in the construction of any new length-biased model is the derivation of its statistical properties. Most papers in the field investigate several of the following features.

5.1 Moments and Shape Characteristics

The derivation of ordinary moments is fundamental because it allows the study of mean, variance, skewness and kurtosis. For many length-biased models, moments are obtained directly from the corresponding moments of the baseline distribution using the weighted-distribution identity. Several studies show that length-biased versions may alter skewness and tail behavior substantially, often producing more appropriate shapes for positively skewed lifetime data.

5.2 Reliability and Hazard Measures

Reliability function, hazard rate, reverse hazard rate, cumulative hazard and mean residual life are among the most frequently derived properties. The hazard function is especially important because one of the advantages of generated lifetime models is their ability to display richer hazard structures.

5.3 Characterization and Stochastic Properties

Some authors have explored characterization results for length-biased families, while others have examined stochastic ordering, preservation of ageing properties and relationships between the original and length-biased models.

6. Estimation and Inference

Parameter estimation and inferential methodology form a central part of the length-biased literature. Maximum likelihood estimation is the most widely used inferential approach in published studies. For models such as the length-biased weighted Lomax, length-biased Sushila, length-biased generalized uniform, length-biased Aradhana and Weibull length-biased exponential distributions, the model parameters are commonly estimated by maximizing the corresponding log-likelihood. Although less common than MLE, moment-based estimation remains useful, particularly as an initial-value strategy in iterative likelihood procedures. In the Weibull length-biased literature, both moment estimators and Bayesian estimators have been discussed.

A foundational contribution to the field is the nonparametric maximum likelihood estimation of the underlying distribution in the presence of length bias. This problem becomes important when one observes a sample from the length-biased distribution. Goodness-of-fit testing under length-biased sampling has received growing attention.

7. Applications

The practical success of length-biased distributions is one of the strongest arguments for their continued development. In reliability engineering, these models are used to study lifetimes of components and systems when longer-lived items are more likely to appear in sampled populations.

In survival analysis and biomedicine, length-biased sampling occurs naturally in cross-sectional and prevalent-cohort studies, where subjects with longer disease duration have a higher chance of being observed.

In ecology and forestry, weighted and size-biased sampling schemes arise when larger organisms or longer measurements are more likely to be included. Length-biased frameworks have therefore been applied to wildlife populations and tree-diameter studies.

In insurance, finance and actuarial science, heavy-tailed and skewed data such as claim sizes or business failure durations can often be better described by length-biased Pareto- or Lomax-type families.

8. Limitations and Research Gaps

Most published work is confined to univariate continuous models. Multivariate length-biased distributions remain underdeveloped, even though many real applications involve dependence among several lifetimes or measurements.

The dominant reliance on maximum likelihood estimation leaves room for broader inferential development. Robust estimation, penalized likelihood, bootstrap-based inference and Bayesian computation deserve greater attention, particularly for small samples and censored observations.

Third, although goodness-of-fit testing has begun to receive attention, model-selection tools designed specifically for length-biased data remain relatively limited. Regression structures for length-biased distributions are scarce.

Finally, many papers focus on deriving numerous mathematical properties for a single new model, but fewer studies compare competing length-biased families in a unified empirical framework.

9. Future Research Directions

The existing literature on length-biased distributions highlights several promising avenues for future research. In particular, extending the framework to multivariate and dependent structures would significantly enhance its applicability in reliability and biomedical studies. Further development of regression-based models, including accelerated failure-time and proportional-hazard formulations under length-biased sampling, is also needed to incorporate covariate effects effectively.

Additionally, the joint treatment of censoring and sampling bias remains a challenging yet important area, especially in survival analysis. Finally, integrating length-biasing with modern distribution-generation techniques and conducting comprehensive comparative studies across models and datasets would contribute to identifying the most suitable approaches for practical applications.

10. Conclusion

Length-biased distributions occupy a central place within the theory of weighted distributions and provide an essential framework for modelling data collected under unequal selection probabilities. Their importance arises not only from elegant mathematical structure but also from their direct practical relevance to lifetime, survival, renewal and reliability data.The reviewed literature shows a clear evolution from foundational theory to a rich collection of specific model families, including Weibull, exponential-generated, Lomax, beta-Pareto, generalized uniform, Sushila, Aradhana and transmuted length-biased distributions.

At the same time, the field still offers substantial opportunities for deeper development. The future of this area will likely depend on stronger inferential tools, multivariate extensions and more systematic application-based comparisons.

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