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Wiener Polynomial of Splitting Graph of Comb Graph

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ABSTRACT

The sum of distances between all unordered pairs of vertices in a connected graph is called the Wiener index. In this work, we found the Wiener polynomial and index of splitting graph of comb graph .

Keywords: Graph theory, Wiener index, Wiener polynomial

1. INTRODUCTION

It is well established that the Wiener index Wexhibits strong correlations with various physicochemical properties of organic compounds, including boiling point, heat of vaporization, heat of formation, chromatographic retention times, surface tension, and vapor pressure. This relationship can be explained by the observation that the Wiener index is approximately proportional to the van der Waals surface area of a molecule.

Since the biological activity of organic compounds is closely related to their molecular structure, it becomes essential to develop suitable mathematical representations of these structures. Such representations are commonly expressed through molecular descriptors, which quantify specific structural features of molecules. Among these, the Wiener index is one of the earliest and most extensively studied topological descriptors, with significant applications in chemistry.

A representation that captures only the number of elements and their connectivity is referred to as a topological representation. In chemistry, this is often modelled using a molecular graph, where atoms are represented as vertices and covalent bonds as edges.

Let Gbe a graph, and let uand v be two vertices of G. If $\mathrm{d(u,v)}$denotes the length of the shortest path between uand v, then the Wiener index of Gis defined as: $\mathrm{W(G)} = \sum_{\mathrm{u,v \in V(G)}} \mathrm{d(u,v)}$,
where the summation is taken over all unordered pairs of vertices in G.

The Wiener polynomial is a related generating function which was first defined by HaruoHosoya[7] with this name, in honor of Harry Wiener but also known today as Hosoya polynomial, extends this concept to capture the complete distribution of distances in graph. If q is a parameter, then the Wiener polynomial of G is $W(G;q) = \sum_{\{u,v\} \subseteq V(G)} q^{d(u,v)}$

This paper gives the expressions for Wiener polynomial of Splitting graph of Comb graph.

2 Splitting graph of a graph: The splitting graph is introduced by Prof . E. Sampath Kumar, Prof. H.B. Walikar in [8]. For a graph G , Splitting graph is denoted by S(G). For each vertex v of a graph G, take a new vertex v' .Join v'

to all the vertices of G adjacent to v. The graph S(G) thus obtained is called Splitting graph or Duplicate graph of G. In [8], they have studied some properties of S(G) and obtained the characterization of S(G). Prof. Jan Mycielski, in his work in sets and logic used a graph called shadow graph S(G). For a graph G , the shadow graph S(G) is obtained from G by adding for each vertex v of G, a new vertex $v^{/}$ called shadow vertex of v, and joining $v^{/}$ to the neighbour of v in G.

2.1 Observations on S(G):

1. A vertex of G and its duplicate vertices are not adjacent in S(G).

2. All duplicate vertices induces a null graph.

3. $d(v_i)\ in\ G\ = d(v_i^{/})\ in\ S(G)$.

4. $d(v_i)\ in\, S(G) = 2d(v_i^{/})\ in\ S(G)$

5. By definition of splitting graph, $d(v_i, v_j / G) = d(v_i, v_j / S(G))$

2.2 Note[9]: As the splitting graph consists of the vertices of given graph and its duplicate vertices, for convenience sake ,partition the vertex set of S(G) into the sets A,B as below.

$$A = \{v_i / v_i \in V(G)\}$$
$$B = \{v_i^{/} / v_i^{/} \in V(S(G)) - V(G)\}$$

In order to demonstrate the validity of the theorem, we adopt the following notation:

$$d_A(G,i) = number\ of\ pairs of\ vertices in\ the set\ A, at\ a\ distance\ i$$
$$d_B(G,i) = number\ of\ pairs of\ vertices in\ the set\ B, at\ a\ distance\ i$$

$$d_{AB}(G,i) = number\ of\ pairs of\ vertices of\ which\ one\ is\ in\ the\ set\ A and\ the\ other\ is\ in\ the set\ B, that\ are\ at\ a\ distance\ i$$

$$d_A(v_i,v_j) = distance\ between\ the\ vertices\ v_i,v_j\ among the\ vertices\ of\ the\ set\ A$$

$$d_B(v_i,v_j) = distance\ between\ the\ vertices\ v_i,v_j\ among the\ vertices\ of\ the\ set\ B$$

$$d_{AB}(v_i,v_j) = distance\ between\ the\ vertices\ v_i,v_j\ among the\ vertices\ of\ the\ set\ A,B$$

With the above notation, the Wiener polynomial of Splitting graph is given as

$$W(S(G,q)) = a_1q + a_2q^2 + a_3q^3 + ... + a_iq^i$$

It can be easily observed that $a_i = d_A(G,i) + d_B(G,i) + d_{AB}(G,i)$

By computing these three terms we found the coeficients of wiener polynomial.

Note: In the following proofs considerj as n whenever $j \equiv 0 (\bmod n)$.

2.3 Theorem : The wiener polynomial of $S(P_n \otimes K_1)$ is

$$W(S(P_n \otimes K_1)) = 4W(P_n \otimes K_1) + (1-3n)q + 2nq^2 + (3n-1)q^3, n \geq 4$$

Proof: Let $v_1, v_2, \ldots v_n \in V(C_n), a_1, a_2, \ldots a_n$ be the pendent vertices that are adjacent to v_i in $(P_n \otimes K_2)$ $v_1^{/}, v_2^{/}, \ldots v_n^{/} \in V(C_n), a_1^{/}, a_2^{/}, \ldots a_n^{/}$ be the corresponding duplicate vertices. For convenience sake we partition the vertex set into the following subsets and found the distances between them.

$$A = Set\,of\ vertices\,of\ comb\,graph$$
$$B = Set\,of\ duplicate\,vertices\,of\ comb\,graph$$

Further the sets A, B are partitioned as

$$A1 = Set\,of\ vertices\,of\ \ path\ of\ comb\,graph\ \ B1 = Set\,of\ duplicatevertices of\ path.$$
$$A2 = Set\,of\ pendent\ vertices\ of comb graph\ \ B2 = Set\,of\ duplicate\ vertices\,of\ \deg ree\ one$$

The distances between the vertices from the above sets are given in ten combitions as follows.

Case1: Wiener polynomial of set of vertices of A1is given in theorem1 as

$$\sum_{\{v_i, v_j\} \in V(A1)} q^{(v_i, v_j)} = (n-1)q^1 + (n-2)q^2 + (n-3)q^3 + \ldots \quad + 1q^{n-1} = \sum_{i=1}^{n-1} (n-i)q^i$$

Case2: Wiener polynomial of set of vertices of A1,A2is given in theorem1 as

$$\sum_{\{v_i, a_j\} \in V(A1A2)} q^{(v_i, a_j)} = nq + 2\left((n-1)q^2 + (n-2)q^3 + \ldots\ldots + 1.q^n\right) = nq + 2\sum_{i=1}^{n-1} (n-i)q^{i+1}$$

Case3: Distance between the vertices of A2 is given as

$$d(a_i, a_j) = j - i + 2\ for 1 \leq i < j \leq n,$$

$$d_{A2}(G, i) = n - i + 2\ for 3 \leq i \leq n+1$$

$$\sum_{\{a_i, a_j\} \in V(A2)} q^{(a_i, a_j)} = (n-1)q^3 + (n-2)q^4 + \ldots\ldots + 1.q^{n+1} = \sum_{i=1}^{n-1} (n-i)q^{i+2}$$

Case4: Distance between the vertices of B1 is given as in theorem1.

Case5: Distance between the vertices of B1,B2 is given as

$$d(v_i^{/}, a_j^{/}) = \begin{cases} 3, for\ i = j \\ |j - i| + 1\ for\ i \neq j, 1 \leq i, j \leq n, \end{cases}$$

$$d_{B1B2}(G, i) = \begin{cases} 3n - 4\ for i = 3 \\ 2(n - i + 1), 2 \leq i \leq n \end{cases}$$

The corresponding wiener polynomial is given as

$$\sum_{\{v_i^{/},a_j^{/}\}\in V(B1B2)} q(v_i^{/},a_j^{/}) = nq^3 + 2\left((n-1)q^2 + (n-2)q^3 + ...+1.q^n\right) = nq^3 + 2\sum_{i=1}^{n-1}(n-i)q^{i+1}$$

Case6:Distance between the vertices of B2 is given as

$$d_{B2}(a_i^{/},a_j^{/}) = j-i+2 \; for \; 1 \le i < j \le n$$

$$d_{B2}(G,i) = (n-i+2), 3 \le i \le n+1$$

$$\sum_{\{a_i^{/},a_j^{/}\}\in V(B2)} q(a_i^{/},a_j^{/}) = (n-1)q^3 + (n-2)q^4 + (n-3)q^5 + ...+1.q^{n+1} = \sum_{i=1}^{n-1}(n-i)q^{i+2}$$

Case7: The distance between the vertices of A1,B1is given as in case – of theorem 1

Case8: Distance between the vertices of A1,B2 is given as

$$d(v_i,a_j^{/}) = |j-i|+1 \; for \; i \ne j, 1 \le i, j \le n,$$

$$d_{A2B1}(G,i) = \begin{cases} n \; for i = 1 \\ 2(n-i+1), 2 \le i \le n \end{cases}$$

$$\sum_{\{v_i,a_j^{/}\}\in V(A2B1)} q(v_i,a_j^{/}) = nq^1 + 2\left((n-1)q^2 + (n-2)q^3 + ...+1.q^n\right) = nq^1 + 2\sum_{i=1}^{n-1}(n-i)q^{i+1}$$

Case9: Distance between the vertices of A2,B1is given as in case5

Case10: Distance between the vertices of A2,B2 is given as

$$d(a_i,a_j^{/}) = |j-i|+2 \; for 1 \le i, j \le n,$$

$$d_{A2B2}(G,i) = \begin{cases} n \; for i = 2 \\ 2(n-i+2), 3 \le i \le n+1 \end{cases}$$

$$\sum_{\{a_i,a_j^{/}\}\in V(A2B2)} q^{(a_i,a_j^{/})} = nq^2 + 2\left((n-1)q3 + (n-2)q^4 + ...+1.q^{n+1}\right) = nq^2 + 2\sum_{i=1}^{n-1}(n-i)q^{i+2}$$

By adding all the above wiener polynomials ,we get

$$W(S(P_n \otimes K_1)) = 4W(P_n \otimes K_1) + (1-3n)q + 2nq^2 + (3n-1)q^3, n \ge 4$$

$$W(S(T_{n,k})) = 4W(T_{n,k}) + (n+k)(q^3 + q^2 - q), n \ge 5$$

Referrences:

[1].A.A Dobrynin, R.Entringer, I. Gutman; Wiener index of trees .Acta Appl. 66(2001)211-249.

[2]. An.X and Wu.B. The Wiener Index of k^{th} power of a graph, Appl.math.Lett.21, 436-440,2008.

[3]. Balakrishnan, Sridharan. R. N and Vishwanathan.kIyer,Wiener Index of graphs with more than one cut vertex, A ppl.Math. Lett 21,922-927,2008.

[4].Bondy, J.A, and Murthy. U.S.R, Graph Theory with applications .(Macnydmillan,London,1976)

[5]. Entringer . R.C, Jackson, D.E and Snyder,D.A, Distance in graphs, zech.Math.J.26,283-296,1976.

[6]. I. GutCman, I.G. Zenkerich,Wiener index and Vibrational energy, Z. Naturforch.57(2002),824-828.

[7]. H. Hosoya, On Some counting polynomials in chemistry, Disc. Appl. Math.19(1988)239-257

[8] E. Sampathkumar and H. B. Walikar The Karnataka University Journal -Science-Vol. XXV & XXVI (Combined) 1980

[9] Latha Devi puli, Thesis of Visveswaraya Technological University