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Oscillation Analysis of Caputo Fractional Differential Equations with Advanced and Delay Arguments via a Monotonicity Approach

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Abstract: In this paper , we investigate the oscillatory behavior of a class of Caputo fractional differen-tial equations involving both advanced and delayed arguments.By establishing novel monotonicity proper-ties of eventually positive solutions,we derive improved sufficient conditions for the nonexistence of positive monotone solutions.New auxiliary functions are introduced to obtain unified monotonicity results without imposing restrictive assumptions on the coefficient functions,thereby extending and strengthening several existing oscillation criteria.The developed approach yields sharper oscillation conditions for both advanced and ratarded fractional differential equations and is further extended to equations containing simultaneously advanced and delayed arguments.Several illustrative examples are presented to demonstrate the effective-ness of the proposed criteria and to highlight the significant improvement over previously known results.The obtained results contribute to the qualitative theory of fractional differential equations and provide a useful framework for the anaiysis of oscillatory behavior in fractional dynamical systems with deviating arguments.

Keywords: Caputo fractional derivative, Fractional differential equations, Oscillation, Advanced argument, Delay differential equation, Monotonicity, Qualitative analysis.

1 Introduction

Fractional calculus has become an important branch of modern applied mathematics because of its remark-able capability to describe systems possessing memory, hereditary characteristics, and nonlocal interactions. Unlike classical integer-order derivatives, fractional derivatives incorporate the past history of a process, making them highly suitable for modelling complex phenomena arising in viscoelasticity, control theory, fluid mechanics, signal processing, biology, finance, and engineering applications. Among the various frac-tional operators, the Caputo fractional derivative has attracted considerable attention since it allows the use of classical initial conditions while maintaining strong physical interpretations [1, 2, 4, 5, 3, 15].

During the past two decades, the qualitative theory of fractional differential equations has witnessed significant development. In particular, the existence, uniqueness, stability, controllability, and oscillatory behaviour of solutions have been extensively investigated owing to their importance in mathematical mod-elling. Comprehensive treatments of fractional differential equations and their analytical properties can be found in the monographs of Podlubny [1], Kilbas *et al.* [2], Diethelm [3], Zhou [15], and Baleanu *et al.* [10]. The theory of fractional dynamic systems has also been systematically developed by Lakshmikantham *et al.* [6].

Oscillation theory plays a fundamental role in understanding the qualitative behaviour of differential equations without requiring explicit analytical solutions. Oscillation criteria provide valuable information regarding the long-term dynamics, stability, persistence, and periodic behaviour of mathematical models. For classical differential equations with delay and advanced arguments, important contributions have been made by Agarwal *et al.* [7], Koplatadze and Chanturia [11], Agarwal, Bohner and Grace [12], and Li and Agarwal [13], who established several sufficient conditions for oscillation and nonoscillation of functional differential equations.

Recently, these ideas have been extended to the fractional-order setting. Grace *et al.* [8] established oscillation criteria for fractional differential equations by employing fractional integral inequalities and com-parison techniques. Chen *et al.* [9] further investigated oscillation conditions for nonlinear fractional dif-ferential equations with damping effects, while Liang and Qiu [14] developed oscillation criteria for Caputo fractional delay differential equations using suitable integral estimates. These studies demonstrate that the

introduction of fractional derivatives considerably enriches the qualitative behaviour of differential equations and simultaneously presents new analytical challenges.

Motivated by these developments, the present paper investigates the oscillatory behaviour of a class of Caputo fractional differential equations involving advanced and delayed arguments. By establishing new monotonicity properties of eventually positive solutions, we derive improved oscillation criteria that substan-tially weaken several existing assumptions available in the literature. New auxiliary functions are introduced to obtain unified monotonicity results for both advanced and retarded fractional differential equations, leading to sharper sufficient conditions for the nonexistence of eventually positive monotone solutions. Furthermore, the proposed approach is extended to fractional differential equations containing simultaneously advanced and delayed arguments. Several illustrative examples are presented to demonstrate the applicability and effectiveness of the obtained criteria and to compare the new results with existing oscillation conditions.

The analytical techniques employed in this work are inspired by modern developments in fractional calculus and qualitative analysis [1, 2, 3, 10, 15], while the motivation for studying oscillatory behaviour

originates from the classical and fractional oscillation literature [7, 8, 9, 11, 12, 13, 14]. In addition, the present investigation complements recent analytical studies on fractional differential equations solved through the Laplace–Adomian Decomposition Method and related analytical techniques [16]. Consequently, the obtained results enrich the qualitative theory of fractional differential equations and provide a useful framework for future investigations on nonlinear fractional dynamical systems with more general deviating arguments.

2 Preliminaries

In this paper we study the asymptotic and oscillation behavior of solutions of Caputo fractional delay differential equation,

$$^{c}D^{\alpha}y(t) = {}^{t}p(t)y(\tau(t)), \qquad 1 < \alpha \le 2 \tag{1}$$

We shall asume that

$$(H_1)\, p(t) \in C^1([t_0,\infty)), \qquad p(t) > 0, \tag{2}$$

$$(H_2)\, \tau(t) \in C^1([t_0,\infty)), \qquad \tau'(t) > 0, \qquad \lim_{t\to\infty} \tau(t) = \infty. \tag{3}$$

By a solution of equation (1) we mean a function $y(t) \in C^2([T_y,\infty))$, $T_y \ge t_0$ which satisfies equation (1) on $[T_y,\infty)$.We consider only those solutions y(t) of (1) for which $Sup\{|y(t)| : t \ge T\} > 0$ for all $T \ge T_y$.

The set of all positive solutions is

$$N = N_0 \bigcup N_2, \tag{4}$$

Where the class N_0 involves positive decreasing solutions while N_2 includes positive increasing ones. If $\tau(t) \le t$ the condition

$$\lim_{t\to\infty} Sup \int_{\tau(t)}^{t} (s-\tau(t))^{\alpha-1} p(s)ds > 1 \tag{5}$$

does not allow the presence of positive decreasing solutions i.e. $N_0 = \phi$. If $\tau(t) \ge t$ the condition

$$\lim_{t\to\infty} Sup \int_{t}^{\tau(t)} (\tau(t)-s)^{\alpha-1} p(s)ds > 1 \tag{6}$$

does not allow the presence of positive increasing solutions i.e. $N_2 = \phi$.

The corresponding comparison equations in the fractional setting are

$$^{C}D^{\alpha}_{t}y(t) = p_*{}^{\alpha}y(t \pm 2\tau_*), \qquad 1 < \alpha \le 2 \tag{7}$$

and

$$^{c}D^{\alpha}_{t}y(t) = \frac{p_*}{t^{\alpha}} y(\lambda t), \quad 1 < \alpha \le 2 \tag{8}$$

3 Main Results

For positive increasing solution of (1) we can improve the monotonicity as follows.

Lemma 1.
If

$$\int_{t_0}^{\infty} sp(s)ds = \infty \tag{9}$$

Where $0 < \alpha \leq 1$ and Assume that $y(t) \in N_2$ then

$$\frac{y(t)}{t} \text{ is increasing } \quad i.e. \frac{y(t)}{t^\alpha} \uparrow \tag{10}$$

Proof:
Assume that y(t) is positive increasing solution of (1) Now we have

$${}^{c}D_t^{\alpha}\left(t^{2\alpha}\, {}^{c}D_t^{\alpha}\frac{y(t)}{t^{\alpha}}\right) = t^{\alpha}p(t)y(\tau(t)) \tag{11}$$

An fractional integration of (11) from t_0 to t in view of (9) yields.

$$t^{2\alpha}\, {}_{c}D_t^{\alpha}\frac{y(t)}{t^{\alpha}} = k + \int_{t_0}^{t} s^{\alpha} p(s)y(\tau(s))ds$$

Now we have given that,

N_2 contains positive increasing solutions. Therefore y(t) becomes positive and increasing. Also we have given that,
$p(t) \geq 0$ and $y(\tau(t)) > 0$

As y(t) is increasing then $y(\tau(s)) \geq y(\tau(t_0)) > 0$ Thus we get,

$$t^{2\alpha}\, {}_{c}D_t^{\alpha}\frac{y(t)}{t^{\alpha}} \geq k + y(\tau(t_0))\int_{t_0}^{t} s^{\alpha} p(s)ds$$

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From our assumption,

$$\int_{t_0}^{\infty} s^{\alpha} p(s)ds = \infty$$

Hence we get,

$$t^{2\alpha}\, {}^{c}D_t^{\alpha} \frac{y(t)}{t^{\alpha}} \to \infty \text{ as } t \to \infty$$

Hence

$$t^{2\alpha}\, {}^{c}_{t}D^{\alpha} \frac{y(t)}{t^{\alpha}} > 0.$$

Since $t^{2\alpha} > 0$

Therefore ${}^{c}_{t}D^{\alpha} \frac{y(t)}{t^{\alpha}} > 0.$/

Hence, $\frac{y(t)}{t^{\alpha}}$ is increasing.

To simplify our notation we introduce the following triple of functions, So

$$\alpha_2(t) = \frac{p'(t)}{p(t)} + \frac{\tau'(t)}{\tau(t)\Gamma(\alpha)} \int_t^{\tau(t)} (\tau(t) - s)^{\alpha-1} p(s)\tau(s)ds,$$

$${}^{c}D_t^{\alpha}\beta_2(t) = \alpha_2(t) \,,$$

$$\gamma_2(t) = p(t)e^{-\beta_2(t)}.$$

Lemma 2.

If

$$\int_{t_0}^{\infty} sp(s)ds = \infty \qquad (12)$$

and $\tau(t) \geq t$. Assume that $y(t) \in N_2$ then $\gamma_2(t)y(\tau(t)) \uparrow$.

Proof:

Assume that (1) possesses a positive increasing solution y(t) i.e. $y(t) \in N_2$. From the assumption p(t), $\tau(t) \in$ $c'([t_0, \infty])$, we apply the Caputo fractional derivative to the (1), which leads to,

$$ {}^{c}D_t^{\alpha} y''(t) = p'(t)y(\tau(t)) + p(t)\tau'(t)y'(\tau(t))$$

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(13) on the other hand, integration (1) from t to $\tau(t)$ we are lead to

$$y'(\tau(t)) = y'(t) + \frac{1}{\Gamma(\alpha)}\int_t^{\tau(t)} (\tau(t)-s)^{\alpha-1} y''(s)ds$$

But $y''(s) = p(s)y(\tau(s))$

$$y'(\tau(t)) = y'(t) + \frac{1}{\Gamma(\alpha)}\int_t^{\tau(t)} (\tau(t)-s)^{\alpha-1} p(s)y(\tau(s))ds$$

Now here it is given that, y(t) is increasing.
Therefore $\tau(s) \geq \tau(t)$.
Hence

$$y(\tau(s)) \geq y(\tau(t))$$

$$y'(\tau(t)) \geq \frac{y(\tau(t))}{\Gamma(\alpha)}\int_t^{\tau(t)} (\tau(t)-s)^{\alpha-1} p(s)ds$$

Hence (12) becomes,

$$ {}^{C}D_t^{\alpha} y''(t) \geq p'(t)y(\tau(t)) + p(t)\tau'(t)\frac{y(\tau(t))}{\Gamma(\alpha)}\int_t^{\tau(t)} (\tau(t)-s)^{\alpha-1} p(s)ds$$

$$ {}^{C}D_t^{\alpha} y''(t) \geq y(\tau(t)) \left[p'(t) + \frac{p(t)\tau'(t)}{\Gamma(\alpha)}\int_t^{\tau(t)} (\tau(t)-s)^{\alpha-1} p(s)ds \right] \quad (14)$$

But we know that from (1)

$$y(\tau(t)) = \frac{y''(t)}{p(t)}$$

$$ {}^{C}D_t^{\alpha} y''(t) \geq y''(t)\left[\frac{p'(t)}{p(t)} + \frac{\tau'(t)}{\Gamma(\alpha)}\int_t^{\tau(t)} (\tau(t)-s)^{\alpha-1} p(s)ds \right]$$

Hence we get,

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$p(t) +$ $\frac{\tau(t)}{\Gamma(\alpha)}$ $(\tau(t) - s)^{\alpha-1} p(s)\tau(s)ds$
t

${}^{c}D^{\alpha}y''(t) \geq \alpha_2(t)y''(t)$
Multiplying by $e^{-\beta 2(t)}$
${}^{c}D^{t}_{\alpha}(e^{-\beta 2(t)}y''(t)) \geq 0$ It gives that,

$e^{-\beta 2(t)}y''(t) \geq 0$

But $y''(t) = p(t)y(\tau(t))$

$e^{-\beta 2(t)}p(t)y(\tau(t)) \geq 0$

$\gamma_2(t)y(\tau(t)) \geq 0$

$\gamma_2(t)y(\tau(t))$ is increasing.

Theorem 1.

Let $\tau(t) \geq t$ and assume that

$$\int_{t_0}^{\infty} sp(s)ds = \infty \qquad (15)$$

.If

$\lim_{t\to\infty} \sup \gamma_2(t)$ th en N_2 = ϕ fo r the fractional system (1). $\tau(t)$

Proof: t

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$$\frac{p(s)(\tau(t)-s)^{\alpha}}{\Gamma(\alpha+1)\gamma_2(s)} ds > 1 \qquad (16)$$

We argue by contradiction.Assume that (1) possesses an eventually positive increasing solution y(t). By using Caputo fractional derivative of order $\alpha \in (0, 1]$.

Integrate (1) from t to u we get,

$$ {}^{c}D_t^{\alpha} y(u) = {}^{c}D_t^{\alpha} y(t) + \int_t^u p(s) y(\tau(s)) ds $$

We know that,

$$ \gamma_2(s) y(\tau(s)) \geq \gamma_2(t) y(\tau(t)) $$

Hence

$$ y(\tau(s)) \geq \frac{\gamma_2(t)}{\gamma_2(s)} y(\tau(t)) $$

Hence the integral becomes

$$ {}^{c}D_t^{\alpha} y(u) \geq \gamma_2(t) y(\tau(t)) \int_t^u \frac{p(s)}{\gamma_2(s)} ds $$

Now apply fractional integral of order α from t to u then we get

$$ y(u) \geq \gamma_2(t) y(\tau(t)) \int_t^u \int_t^x \frac{p(s)}{\gamma_2(s)} \frac{(x-s)^{\alpha-1}}{\Gamma(\alpha)} ds\, dx $$

Now we change the order of integration then we get

$$ y(u) \geq \gamma_2(t) y(\tau(t)) \int_t^u \frac{p(s)}{\gamma_2(s)} \frac{(u-s)^{\alpha}}{\Gamma(\alpha+1)} ds $$

Setting $u = \tau(t)$ we have

$$ y(\tau(t)) \geq y(\tau(t)) \gamma_2(t) \int_t^{\tau(t)} \frac{p(s)(\tau(t)-s)^{\alpha}}{\Gamma(\alpha+1)\gamma_2(s)} $$

ds

Which contradicts (14) and we conclude that class N_2 is empty.
The above theorem can be reformulated in term of unbounded oscillation.

Theorem 2.
Let $\tau(t) \geq t$ and (9),(14) hold. Then every unbounded solution of (1) is oscillatory.

If $1/\gamma_2(t)$ is increasing, then it is easy to see that (6) implies (14). In the following couple of illustra-tive examples we will demonstrate the real progress.

Example 1.

We consider the advanced fractional differential equation with constant coefficients

$$^{C}D_t^{\alpha} y(t) = p_*^2 y(t + 2\tau_*), \; p_* > 0, \; \tau_* > 0 \tag{17}$$

It is easy to verify that criterion (6) guarantees $N_2 = \phi$ provided that

$$p_*\tau_* > \frac{1}{\sqrt{2}} \approx 0.707106.$$

On the other hand, simple calculation yields.

$$\alpha_2(t) = 2p_*^2\tau_* + \frac{2p_*^2\tau_*^2}{\Gamma(\alpha)}(t + 2\tau_*)^{\alpha-1}$$

Thus,

$$\beta_2(t) = 2p_*^2\tau_* t + \frac{2p_*^2\tau_*^2}{\Gamma(\alpha+1)}(t + 2\tau_*)^{\alpha}$$

and

$$\gamma_2(t) = p_*^2 exp\left(-2p_*^2\tau_* t - \frac{2p_*^2\tau_*^2}{\Gamma(\alpha+1)}(t + 2\tau_*)^{\alpha}\right)$$

and we conclude that $\gamma_2(t)y(t + 2\tau_*)$ is increasing. But then also

$$\bar{\gamma}_2(t)y(t + 2\tau_*) \uparrow \text{with } \bar{\gamma}_2(t) = e^{-2p_*^2\tau_* t}.$$

The reason for introducing function $\bar{\gamma}_2(t)$ is the fact that it is more acceptable than $\alpha_2(t)$ because the fractional term $exp\left(-2p_*^2\tau_*^2(t + 2\tau_*)^{\alpha}\right)$ does not bring any progress for monotonicity of $y(t) \in N_2$.

$\Gamma(\alpha+1)$ $*$ 2

Condition (14) with $\gamma_2(t) = \bar{\gamma}_2(t)$ reduces to

$$e^{4p^2\tau^2} \quad 2 \quad 2 \quad * \quad * > 8p_*\tau_* + 1 \tag{18}$$

By standard numerical methods we can verify that the inequality

$$e^x > 2x + 1 \quad , \qquad x > 0$$

it satisfied for all $x > 1.256431$ and consequently (16) holds true provided that

$$p_*\tau_* > \sqrt{\frac{1.256431}{4}} \approx 0.560453$$

Consequently, our progress is significant.

Example 2.

We consider the advanced fractional differential equation

$$^{C}D^{\alpha}y(t) = \frac{p_*}{t^{\alpha}}\, y(\lambda t) \quad , \qquad 0 < \alpha \le 1 \quad t \tag{19}$$

With $p_* > 0$ and $\lambda > 1$. According to fractional analogue of Koplatadze and Chanturia's criterion,(6) the class $N_2 = \phi$ for equation (17) provided that

$$p^* \left(\frac{\lambda^{\alpha} - 1}{\alpha} - In\lambda \right) > 1 \tag{20}$$

On the other hand, simple calculation yields.

$$\alpha_2(t) = \frac{-\alpha + p_* \lambda^{\alpha} ln\lambda}{t}$$

Therefore,

$\beta_2(t) = (-\alpha + p_* \lambda^\alpha In\lambda)$ In t, and $\gamma_2(t) = t^{-p_* \lambda^\alpha In\lambda}$

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Now, some necessary calculation yields that (14) takes the form

$$p_* \frac{\lambda p_* \lambda^\alpha In\lambda - \lambda^\alpha}{p_* \lambda^\alpha In\lambda - \alpha} - 1 \frac{\lambda p_* \lambda^\alpha In\lambda \; -}{p_* \lambda^\alpha In\lambda} > 1 \tag{21}$$

To check the progress,

Let $\lambda = e$ then fractional condition becomes,

$$p^* \frac{e^\alpha - 1}{\alpha} - 1 > 1$$

Hence

$$p_* > \frac{1}{e^{\alpha} - 1} = \frac{1}{\alpha}$$

then the above inequality reduces to

$$\frac{(e^{p_* e^\alpha} - e^\alpha)p_* e^\alpha - (e^{p_* e^\alpha} - 1)(p_* e^\alpha - \alpha)}{p_* e^\alpha - \alpha} > e^\alpha \tag{22}$$

It is easy to verify that the inequality,

$$\frac{(e^x - e^\alpha)x - (e^x - 1)(x - \alpha)}{x - \alpha} > e^\alpha$$

For $\alpha = 1$, this reduces exactly to original result.

$$p_* > \frac{1.911063}{e} \approx 0.703041$$

Our progress is outstanding.

The next consideration are intended to improve the fractional criterion criterion (5). We define triple of function as follows

$$\alpha_0(t) = - \frac{p'(t)}{p(t)}$$

t

$+ \tau'(t)$ $\int$ $p(s)ds$ (23)

$\tau(t)$

$$\beta_0'(t) = \alpha_0(t) \quad (24)$$

$$\gamma_0(t) = p(t)e^{\beta_0(t)} \quad (25)$$

Lemma 3.

If $\tau(t) \leq t$ and assume that $y(t) \in N_0$ then $\gamma_0(t)y(\tau(t)) \downarrow$.

Proof:

Assume that (1) possesses a positive decreasing solution y(t),i.e $y(t) \in N_0$.Proceeding exactly as in the proof of Theorem 2 we are led to (12).On the other hand,an integration (1) from $\tau(t)$ to t yields

$$-{}^{c}D^{\alpha}_{t}y(\tau(t)) = -{}^{c}D^{\alpha}_{t}y(t) + \int_{\tau(t)}^{t} p(s)y(\tau(s))ds \geq y(\tau(t))\int_{\tau(t)}^{t} p(s)ds \tag{26}$$

We conclude

$${}^{c}D^{\alpha+1}_{t}y(t) \leq y(\tau(t))\left(p'(t) - p(t)\tau'(t)\int_{\tau(t)}^{t} p(s)ds\right)$$

by combining inequalities (12) and (24).Taking (1) into account,we obtain

$${}^{c}D^{\alpha+1}_{t}y(t) + \alpha_0(t){}^{c}D^{\alpha}_{t}y(t) \leq 0$$

Consequently,

$$(e^{\beta_0(t)}\,{}^{c}_{t}D^{\alpha}y(t))' \leq 0$$

and one can see that $e^{\beta_0(t)}\,{}^{c}D^{\alpha}$ is decreasing,which in view of (1) equivalent to $\gamma_0(t)y(\tau(t))$ is decreas-
ing.

Theorem 3.

Let $\tau(t) \leq t$.
If

$$\limsup_{t\to\infty} \gamma_0(t)\int_{\tau(t)}^{t} \frac{p(s)(s-\tau(t))^{\alpha}}{\Gamma(\alpha+1)\gamma_0(s)}ds > 1 \tag{27}$$

then $N_0 = \phi$ for (1).

Proof:

We admit that $N_0 \neq \phi$ which means that (1) posseses an eventually positive decreasing solution y(t). Integrating (1) from u to t and using the monotonicity of $\gamma_0(t)y(\tau(t))$, we obtain

$\gamma_0(s)$ $ds.$

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$$-{}^{C}D_t^{\alpha} y(u) = -{}^{C}D_t^{\alpha} y(t) + \int_u^t p(s)y(\tau(s))ds \geq \gamma_0(t)y(\tau(t)) \int_u^t p(s)$$

Integrating once more that means of the fractional integral operator, we have

$$y(u) \geq \gamma_0(t)y(\tau(t)) \int_u^t \int_x^t \frac{p(s)}{\gamma_0(s)} \frac{(s-x)^{\alpha-1}}{\Gamma(\alpha)} ds\,dx$$

By changing the order of integration we get

$$y(u) \geq \gamma_0(t)y(\tau(t)) \int_u^t \frac{p(s)(s-u)^{\alpha}}{\Gamma(\alpha+1)\gamma_0(s)} ds.$$

Setting $u = \tau(t)$, we have

$$y(\tau(t)) \geq y(\tau(t))\gamma_0(t) \int_{\tau(t)}^t \frac{p(s)(s-\tau(t))^{\alpha}}{\Gamma(\alpha+1)\gamma_0(s)} ds,$$

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$\tau(t)$

Which contradicts to condition (25) and we deduce that class N_0 is empty.

Theorem 4.

If $\tau(t) \le t$ and (25) hold then every bounded solution of (1) is oscillatory. We again support our results by couple of illustrative examples.

Example 3.

We consider the retarded Caputo fractional differential equation of order $0 < \alpha \le 1$ with constant coefficients

$$^{C}D_t^{\alpha} y(t) = p_*^2 y(t - 2\tau_*) \quad , p_* > 0, \tau_* > 0 \tag{28}$$

By (5) the class $N_0 = \phi$ provided that

$$p_*\tau_*^{\alpha} > \frac{1}{\sqrt{2}} \approx 0.707106.$$

On the other hand,

$$\alpha_0(t) = 2p_*^2\tau_*^{\alpha},$$

$$\beta_0(t) = 2p_*^2\tau_*^{\alpha}t^{\alpha},$$

$$\gamma_0(t) = p_*^2 e^{2p_*^2\tau_*^{\alpha}t^{\alpha}}.$$

Consequently condition (25) reduces to

$e^{4p^2\tau^{2\alpha}}$ $\qquad 2 \quad 2\alpha$

$* \ * \ > 8p_*\tau_* + 1.$ (29)

Proceeding exactly as in Example 1 we see that (27) holds if $\sqrt[q]{p_*\tau_4^\alpha} > \frac{1.256431}{} \approx 0.560453.$

Example 4.

Consider the fractional delay Euler differential equation

$$ {}^{C}_{t}D^{\alpha}y(t) = \frac{p_*}{t^\alpha}y(\lambda t), \qquad (30)$$

with $p_* > 0$ and $\lambda \in (0, 1)$.By (5) class $N_0 = \phi$ for Eq.(28) provided that

$$p_*(\lambda^\alpha - 1 - \alpha In\lambda) > 1$$

Which for $\lambda = 1/e$ takes the form

$$p_* > e = 2.718281.$$

On the other hand,

$$\alpha_0(t) = \frac{\alpha + p_*(1-\lambda^\alpha)}{t^\alpha},$$

$$\beta_0(t) = (\alpha + p_*(1-\lambda^\alpha))Int,$$

$\gamma_0(t) = p_* t^{p_*(1-\lambda^\alpha)}$. Now (25) takes the form

$$p_* \lambda^{-p_*(1-\lambda^\alpha)} \frac{1}{p_*(1-\lambda^\alpha)} - \frac{\lambda^{-p_*(1-\lambda^\alpha)} - \lambda^\alpha}{1+p_*(1-\lambda^\alpha)} > 1.$$

Which is equivalent to

$$\lambda^{-p_*(1-\lambda^\alpha)} > 2 + 2p_*(1-\lambda^\alpha) - 2\lambda^\alpha p_*(1-\lambda^\alpha) - \lambda^\alpha. \qquad (31)$$

We denote $x = p_*(1-\lambda^\alpha)$ and set $\lambda = 1/e$. Then (29) becomes algebraic inequality

$$e^x > 2 + 2x - 2e^{-\alpha}x - e^{-\alpha} \quad , \quad x > 0 \ , \ \alpha = 1.$$

Which holds true for all $x > 1.110604$ and so we deduce that

$$p_* > \frac{1.110604}{1 - e^{-\alpha}} \approx 1.756950.$$

4 Extension

From our previous results it is natural to expect that there will be no nonoscillatory solution, or equivalently all solutions will be oscillatory for certain fractional differential equations involving both advanced and de-layed arguments. The purpose of this section is to show that this is indeed the case for

$$^{c}D^{\alpha}y(t) = p(t)y(\tau(t)) + q(t)y(\sigma(t)) \qquad (32)$$

Where p(t),q(t),$\tau(t)$,$\sigma(t)$ obey the corresponding conditions presented in (H_1) and (H_2), and $\gamma_0(t)$ is defined by (21),(22),(23) with p(t) and $\tau(t)$ replaced by q(t) and $\sigma(t)$, respectively.

Theorem 5.

Consider the mixed Caputo fractional differential equation

$$^{c}D^{\alpha}y(t) = p(t)y(\tau(t)) + q(t)y(\sigma(t)), \qquad 0 < \alpha \leq 1, \qquad (33)$$

where,

$$\tau(t) \geq t, \qquad \sigma(t) \leq t,$$

and suppose that (9) and (14) holds. Moreover, assume that

$$\limsup_{t\to\infty} \gamma_0(t) \int_{\sigma(t)}^{t} \frac{q(s)(s-\sigma(t))^{\alpha}}{\Gamma(\alpha+1)\gamma_0(s)}\, ds > 1.$$

Then every solution of equation (32) is oscillatory.

proof:

Assume, to the contrary, that equation (32) possesses a nonoscillatory solution $y(t)$. Without loss of gener-ality, suppose that

$y(t) > 0, \quad t \geq T,$

for some sufficiently large $T > 0$.

Since $y(t)$ is eventually positive, there are only two possible cases.

Case 1. Assume that $y(t)$ is eventually increasing. Since

$$q(t)y(\sigma(t)) \geq 0,$$

equation (32) gives

$$^{C}D^{\alpha}_{t}y(t) = p(t)y(\tau(t)) + q(t)y(\sigma(t)) \geq p(t)y(\tau(t)).$$

Hence every eventually positive increasing solution of (32) is also a supersolution of the advanced fractional equation

$$^{C}D^{\alpha}_{t}y(t) = p(t)y(\tau(t)).$$

By Lemma 2,

$$\gamma_2(t)y(\tau(t))$$

is increasing.

Proceeding exactly as in the proof of Theorem 1 and applying the fractional integral operator, we obtain

$$y(\tau(t)) \geq \gamma_2(t)y(\tau(t)) \int_t^{\tau(t)} \frac{p(s)(\tau(t)-s)^{\alpha}}{\Gamma(\alpha+1)\gamma_2(s)} ds.$$

Dividing both sides by the positive quantity $y(\tau(t))$ yields

$$1 \geq \gamma_2(t) \int_t^{\tau(t)} \frac{p(s)(\tau(t)-s)^{\alpha}}{\Gamma(\alpha+1)\gamma_2(s)} ds.$$

Taking the limit superior as $t \to \infty$,

$$1 \geq \limsup \gamma_2(t) \int$$

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$ds,$

$\Gamma(\alpha+1)\gamma(s)$

$^{\tau(t)}\ p(s)(\tau(t)-s)^{\alpha}$

$t\to\infty \quad t \quad 2$

which contradicts condition (16).
Therefore, equation (32) admits no eventually positive increasing solution.

Case 2. Assume that $y(t)$ is eventually decreasing. Since

$$p(t)y(\tau(t)) \ge 0,$$

equation (32) implies

$$^{C}D_t^{\alpha}y(t) = p(t)y(\tau(t)) + q(t)y(\sigma(t)) \ge q(t)y(\sigma(t)).$$

Thus every eventually positive decreasing solution satisfies the delay fractional inequality. By Lemma 3,

$\gamma_0(t)y(\sigma(t))$

is
decreasing.
Proceeding exactly as in the proof of Theorem 3, we obtain

$$y(\sigma(t)) \ge \int_{\sigma(t)}^{t} \frac{q(s)(s-\sigma(t))^{\alpha}}{\Gamma(\alpha+1)\gamma_0(s)}\,ds.$$
$$\gamma_0(t)y(\sigma(t))$$

Dividing both sides by the positive quantity $y(\sigma(t))$ gives

$$1 \geq \gamma_0(t) \int_{\sigma(t)}^{t} \frac{q(s)(s-\sigma(t))^{\alpha}}{\Gamma(\alpha+1)\gamma_0(s)} ds.$$

Taking the limit superior,

$$1 \geq \limsup_{t\to\infty} \gamma_0(t) \int_{\sigma(t)}^{t} \frac{q(s)(s-\sigma(t))^{\alpha}}{\Gamma(\alpha+1)\gamma_0(s)} ds,$$

which contradicts the assumed hypothesis.
Hence no eventually positive decreasing solution exists.
Since every nonoscillatory solution must eventually be either increasing or decreasing, both possibilities lead to contradictions.
Therefore, equation (32) possesses no nonoscillatory solutions. Hence every solution of equation (32) is oscillatory.

5 Conclusion

In this paper, we investigated the oscillatory behavior of a class of Caputo fractional differential equations involving advanced and delayed arguments. By establishing new monotonicity properties for eventually positive solutions, we derived improved oscillation criteria for both advanced and retarded fractional dif-ferential equations. The obtained sufficient conditions exclude the existence of positive monotone solutions under weaker assumptions, thereby extending and strengthening several previously known oscillation results. Furthermore, the developed theory was extended to mixed fractional differential equations containing both advanced and delayed arguments. The illustrative examples confirmed the effectiveness and applicability of the proposed criteria and demonstrated a significant improvement over existing results. Therefore, the present work contributes to the qualitative theory of fractional differential equations and provides a useful framework for future research on nonlinear fractional models, variable-order fractional systems, and more general equations with deviating arguments.

References

[1] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[2] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, 2006.

[3] K. Diethelm, *The Analysis of Fractional Differential Equations*, Springer, Berlin, 2010.

[4] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach Science Publishers, New York, 1993.

[5] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equa-tions*, John Wiley & Sons, New York, 1993.

[6] V. Lakshmikantham, S. Leela and J. Vasundhara Devi, *Theory of Fractional Dynamic Systems*, Cam-bridge Scientific Publishers, 2009.

[7] R. P. Agarwal, S. R. Grace and D. O'Regan, Oscillation Theory for Difference and Functional Differential Equations, Kluwer Academic Publishers, Dordrecht, 2000.

[8] S. R. Grace, R. P. Agarwal, P. J. Y. Wong and A. Zafer, On the oscillation of fractional differential equations, *Fractional Calculus and Applied Analysis*, 13(2010), 225–236.

[9] D. X. Chen, J. S. Yu and P. Li, Oscillation criteria for nonlinear fractional differential equations with damping, *Electronic Journal of Differential Equations*, 2013(2013), No. 59, 1–10.

[10] D. Baleanu, K. Diethelm, E. Scalas and J. J. Trujillo, *Fractional Calculus: Models and Numerical Methods*, World Scientific, Singapore, 2012.

[11] R. Koplatadze and T. Chanturia, Oscillating and monotone solutions of first-order differential equations with deviating arguments, *Differential Equations*, 18(1982), 1463–1465.

[12] R. P. Agarwal, M. Bohner and S. R. Grace, Oscillation criteria for delay differential equations, *Computers and Mathematics with Applications*, 53(2007), 1573–1580.

[13] T. Li and R. P. Agarwal, Oscillation criteria for second-order delay differential equations, *Journal of Mathematical Analysis and Applications*, 331(2007), 1024–1037.

[14] J. Liang and T. Qiu, Oscillation of Caputo fractional differential equations with delay, *Advances in Difference Equations*, 2015:298, 1–15.

[15] Y. Zhou, *Basic Theory of Fractional Differential Equations*, World Scientific Publishing, Singapore, 2014.

[16] Jyoti Thenge-Mashale, Vishal Koli and Amruta Londhe, *Analytical Solutions of Fractional Riccati Dif-ferential Equations via the Laplace Adomian Decomposition Method*, IJSREM,05, 2026.