International Journal of Science, Strategic Management and Technology
Volume **02** Issue **08** August-2026 | ISSN: **3108-1762 (Online)** | Impact Factor: 3.8
An International, Peer-Reviewed, Open Access Scholarly Journal Indexed in recognized academic databases

Upper Bounds for the Product Harary–Furtula Index of Splice and Thorn Graphs

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https://doi.org/ 10.55041/ijsmt.v2i8.058

Cite this Article: Kandan, P. & Niranjan, S. (2026). Upper Bounds for the Product Harary–Furtula Index of Splice and Thorn Graphs. International Journal of Science, Strategic Management and Technology, *02*(8), 1-9. https://doi.org/10.55041/ijsmt.v2i8.058

Abstract

This paper studies upper bounds for the product Harary–Furtula index $RD^*_{HF}(G)$ of splice and thorn graphs. For the splice graph obtained by joining selected vertices of two connected graphs with a new edge, the changes in vertex degrees and distances are analyzed to derive an upper bound in terms of the structural parameters of the constituent graphs. The study is further extended to thorn graphs formed by attaching pendant vertices to the vertices of a given graph. Suitable parameters involving the number of pendant vertices, degree bounds, and distance contributions are introduced to establish an upper bound for the product Harary–Furtula index. A special case in which the same number of pendant vertices is attached to every vertex is also considered. The results provide useful inequalities for the product Harary–Furtula index under these graph constructions.

MSC: 05C12, 05C76

Keywords: Product Harary–Furtula Index, Splice Graph, Thorn Graph, Graph Operations, Upper Bounds.

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1 Introduction

All graphs considered in this paper are simple and connected. The degree of a vertex x in a graph G is denoted by $d_G(x)$, and the distance between two vertices x and y is denoted by $d_G(x,y)$. Graph invariants provide useful mathematical descriptors for investigating structural properties of graphs and molecular structures. In chemical graph theory, such invariants are commonly referred to as topological indices and have been extensively used in molecular modeling, quantitative structure–property relationship (QSPR) studies, and structure–activity investigations [1, 2]. Among the different classes of topological indices, distance-based indices have received considerable attention because they incorporate the distances between pairs of vertices.

The Wiener index is one of the earliest and most extensively studied distance-based topological indices. It was introduced by Wiener in 1947 in connection with the study of the boiling points of paraffin compounds and is defined by $W(G) = \sum_{\{x,y\}\subseteq V(G)} d_G(x,y)$. The reciprocal counterpart of the Wiener index is the Harary index, defined by $H(G) = \sum_{\{x,y\}\subseteq V(G)}$. The Harary index was introduced in the context of chemical graph theory and has subsequently been investigated for various classes of graphs [3, 4]. Distance-based descriptors such as the Wiener and Harary indices have also been studied in relation to molecular structure and other graph invariants [5].

Degree-based topological indices constitute another important class of graph invariants. Among them, the Zagreb indices introduced by Gutman and Trinajstić have played a fundamental role in mathematical chemistry [6]. The first and second Zagreb indices of a graph G are respectively defined by $M_1(G) = \sum_{uv\in E(G)}(d_G(u) + d_G(v))$ and $M_2(G) = \sum_{uv\in E(G)} d_G(u)d_G(v)$. Another important degree-based descriptor is the Forgotten index, introduced by Furtula and Gutman, which is defined by $F(G) = \sum_{u\in V(G)} d_G^3(u) = \sum_{uv\in E(G)} \left(d_G^2(u) + d_G^2(v)\right)$. These indices have been widely investigated because of their structural interpretations and their relationships with other graph invariants [7, 8].

The interaction between degree and distance information has led to the development of several combined topological descriptors. In particular, the Harary–Gutman and Harary–Furtula indices incorporate vertex degrees into the reciprocal-distance framework. The Harary–Gutman index and Harary–Furtula index are respectively defined by

$$RD^{+}_{HG}(G) = \sum_{\{x,y\}\subseteq V(G)} \frac{d_G(x) + d_G(y)}{d_G(x,y)}$$

and

$$RD^{+}_{HF}(G) = \sum_{\{x,y\}\subseteq V(G)} \frac{d_G^2(x) + d_G^2(y)}{d_G(x,y)}.$$

The product versions of these descriptors are given by

$$RD^{*}_{HG}(G) = \prod_{\{x,y\}\subseteq V(G)} \frac{d_G(x) + d_G(y)}{d_G(x, y)}$$

and

$$RD^{*}_{HF}(G) = \prod_{\{x,y\}\subseteq V(G)} \frac{d_G^2(x) + d_G^2(y)}{d_G(x, y)}.$$

The latter index is the principal object of investigation in the present paper. The product Harary–Furtula index incorporates both degree and distance information and therefore reflects structural properties of a graph through a multiplicative framework [9, 10].

Recent studies on topological indices have considered reciprocal versions, products of topological indices, and relationships between distance- and degree-based descriptors. Such investigations provide a framework for studying multiplicative graph invariants and their extremal properties. The Harary index and its generalizations have also been investigated for different graph classes and graph operations [4, 5]. These developments motivate the study of product versions of degree–distance topological indices for various graph constructions.

Motivated by these developments, the present paper investigates upper bounds for the product Harary–Furtula index for two important graph constructions, namely splice graphs and thorn graphs. First, we consider the splice graph obtained from two connected graphs by joining selected vertices with a new edge. The addition of the connecting edge changes the degrees of the selected vertices and also affects the distances between vertices belonging to the two constituent graphs. By analyzing these degree and distance contributions, an upper bound for $RD^{*}_{HF}(G)$ is established in terms of the structural parameters of the original graphs.

The study is further extended to thorn graphs, which are obtained by attaching pendant vertices to the vertices of a given graph. The addition of pendant vertices produces changes in both degree and distance contributions to the product Harary–Furtula index. To handle these contributions, suitable parameters involving the numbers of pendant vertices, degree bounds, and distance-related quantities are introduced. Using these parameters, an upper bound for $RD^{*}_{HF}(G)$ is derived. A special case in which the same number of pendant vertices is attached to every vertex of the base graph is also considered.

Thus, the main objective of this paper is to establish upper bounds for the product Harary–Furtula index of splice and thorn graphs. The obtained results provide relationships between $RD^{*}_{HF}(G)$ and fundamental structural parameters such as degree, distance, order, size, and the parameters associated with the constituent or base graphs. These results contribute to the study of product degree–distance topological indices and their behavior under graph constructions.

International Journal of Science, Strategic Management and Technology
Volume 02 Issue 08 August-2026 | ISSN: 3108-1762 (Online) | Impact Factor: 3.8
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2 Splice graph

Let $G_i = (s_i, t_i)$ graph $i = 1, 2$. The graph H obtained from G_1 and G_2 by joining an edge x_1y_1, where $x_1 \in V(G_1)$ and $y_1 \in V(G_2)$. The graph H is called splice of G_1 and G_2.

Theorem 1. Let H be a splice graph of G_1 and G_2. Then $RD^{*}_{HF}(H) \leq \left(\frac{P}{s_1+s_2}\right)^{s_1+s_2}$, where $P = RD^{+}_{HF}(G_1) + RD^{+}_{HF}(G_2) + \left(\frac{s_2(\Delta_1^2+\Delta_2^2)+2(4\Delta_1+\Delta_2)+5}{4}\right)\sigma^{-1}_{G_1}(x_1) + \left(\frac{s_1(\Delta_1^2+\Delta_2^2)+2(\Delta_1+4\Delta_2)+5}{5}\right)\sigma^{-1}_{G_2}(y_1) + (\Delta_1^2 + \Delta_2^2) + 2(\Delta_1 + \Delta_2) + 2 + (s_1 - 1)\left(\frac{\Delta_1^2+2\Delta_2+\Delta_2^2+1}{4}\right) + (s_2 - 1)\left(\frac{\Delta_2^2+2\Delta_1+\Delta_1^2+1}{4}\right)^2 + (s_1 - 1)(s_2 - 1)\left(\frac{\Delta_1^2+\Delta_2^2}{4}\right)^2$.

Proof. Let $V(G_1) = \{x_1, x_2, x_3, \cdots, x_{s_1}\}$ and $V(G_2) = \{y_1, y_2, y_3, \cdots, y_{s_2}\}$ The new edge joined between the vertices x_1 and y_1 denoted by $e = x_1y_1$. Then

$$\begin{aligned}
P_1 &= \sum_{j=2}^{s}\frac{(d_{G_1}(x_1)+1)^2 + d^2_{G_1}(x_j)}{d_{G_1}(x_i, x_j)} + \sum_{2\leq i<j\leq s_1}\frac{d^2_{G_1}(x_1) + d^2_{G_1}(x_j)}{d_{G_1}(x_i, x_j)} \\
&= \sum_{j=2}^{s_1}\frac{d^2_{G_1}(x_1) + 1 + 2d_{G_1}(x_1) + d^2_{G_1}(x_j)}{d_{G_1}(x_i, x_j)} + \sum_{2\leq i<j\leq s}\frac{d^2_{G_1}(x_1) + d^2_{G_1}(x_j)}{d_{G_1}(x_i, x_j)} \\
&= RD^{+}_{HF}(G_1) + \sum_{j=2}^{s_1}\frac{1 + 2d_G(x_1)}{d_{G_1}(x_1, x_j)} \\
&\leq RD^{+}_{HF}(G) + \sum_{j=2}^{s_1}\frac{1 + 2\Delta_1}{d_{G_1}(x_1, x_j)} \\
&= RD^{+}_{HF}(G) + (1 + 2\Delta_1)\sigma^{-1}_{G_1}(x_1).
\end{aligned}$$

$$\begin{aligned}
P_2 &= \sum_{i=2}^{s}\frac{(d_{G_2}(y_1)+1)^2 + d^2_{G_2}(y_j)}{d_{G_2}(y_i, y_j)} + \sum_{2\leq i<j\leq s_2}\frac{d^2_{G_2}(y_1) + d^2_{G_2}(y_j)}{d_{G_2}(y_i, y_j)} \\
&\sum_{1\leq i<j\leq s_2}\frac{d^2_{G_2}(y_i) + d^2_{G_2}(y_j)}{d_{G_2}(y_i, y_j)} + \sum_{i=1}^{s_2}\frac{1 + 2d_{G_2}(y_1)}{d_{G_2}(y_i, y_j)} \\
&\leq RD^{+}_{HF}(G_2) + (1 + 2\Delta_2)\sigma^{-1}_{G_2}(y_1).
\end{aligned}$$

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$$
\begin{aligned}
P_3 &= (d_{G_1}(x_1)+1)^2 + (d_{G_1}(y_1)+1)^2 \\
&= \left(d^2_{G_1}(x_1) + d^2_{G_2}(y_1)\right) + 2\left(d_{G_1}(x_1) + d_{G_2}(y_1)\right) + 2 \\
&\le \Delta_1^2 + \Delta_2^2 + 2(\Delta_1+\Delta_2) + 2.
\end{aligned}
$$

$$
\begin{aligned}
P_4 &= \sum_{j=2}^{s_2} \frac{(d_{G_1}(x_1)+1)^2 + d^2_{G_1}(y_j)}{d_{G_2}(y_1,y_j)+1} \\
&= \sum_{j=2}^{s_2} \frac{d^2_{G_1}(x_1) + 1 + 2d_{G_1}(x_1) + d^2_{G_2}(y_j)}{d_{G_2}(y_1,y_j)+1} \\
&\le \sum_{j=2}^{s_2} \frac{\Delta_1^2 + 2\Delta_1 + 1 + \Delta_2^2}{d_{G_2}(x_1,y_j)+1} \\
&\le \frac{1}{4}\sum_{j=2}^{s_2}\left[\frac{(\Delta_1^2 + 2\Delta_1 + \Delta_2^2 + 1)}{d_{G_2}(y_1,y_j)} + (\Delta_1^2 + 2\Delta_1 + 1 + \Delta_2^2)\right] \\
&= \frac{1}{4}(\Delta_1^2 + 2\Delta_1 + \Delta_2^2 + 1)\left[\sigma^{-1}_{G_2}(y_1) + s_2 - 1\right].
\end{aligned}
$$

$$
P_5 = \sum_{i=2}^{s_1} \frac{(d_{G_2}(y_1)+1)^2 + d^2_{G_2}(x_i)}{d_{G_1}(x_1,x_i)+1}.
$$

Since here we find the sum between y_1 to x_i, $1 \le i \le s_1$

$$
\begin{aligned}
P_5 &= \sum_{i=2}^{s_1} \frac{d^2_{G_2}(y_1) + 1 + 2d_{G_2}(y_1) + d^2_{G_1}(x_i)}{d_{G_1}(x_1,x_i)+1} \\
&\le \frac{1}{4}\sum_{i=2}^{s_1}\left(\Delta_2^2 + 1 + 2\Delta_2 + \Delta_1^2\right)\left(\frac{1}{d_{G_1}(x_1,x_i)} + 1\right) \\
&\le \frac{1}{4}\left(\Delta_2^2 + 1 + 2\Delta_2 + \Delta_1^2\right)\left(\sigma^{-1}_{G_1}(x_1) + s_1 - 1\right)
\end{aligned}
$$

$$
\begin{aligned}
P_6 &= \sum_{j=2}^{s_2}\sum_{i=2}^{s_1} \frac{d^2_{G_1}(x_i) + d^2_{G_2}(y_j)}{d_{G_1}(x_1,x_i) + d_{G_2}(y_1,y_j) + 1} \\
&\le \frac{1}{4}\sum_{j=2}^{s_2}\sum_{i=2}^{s_1}(\Delta_1^2 + \Delta_2^2)\left(\frac{1}{d_{G_1}(x_1,x_i)} + \frac{1}{d_{G_2}(y_1,y_j)} + 1\right) \\
&= \frac{1}{4}(\Delta_1^2 + \Delta_2^2)\left[(s_2-1)\sigma^{-1}_{G_1}(x_1) + (s_1-1)\sigma^{-1}_{G_2}(y_1) + (s_1-1)(s_2-1)\right].
\end{aligned}
$$

By the definition of RD^*_{HF} for the graph H, we have

$$
RD^*_{HF} \le \left[\left(\frac{P}{s_1+s_2}\right)\right]^{s_1+s_2}
$$

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where

$$
\begin{aligned}
P &= P_1 + P_2 + P_3 + P_4 + P_5 + P_6 \\
&= RD^*_{HF}(G_1) + (1 + 2\Delta_1)\sigma^{-1}_{G_1}(x_1) + RD^*_{HF}(G_2) + (1 + 2\Delta_2)\sigma^{-1}_{G_2}(y_1) \\
&\quad + (\Delta_1^2 + \Delta_2^2) + 2(\Delta_1 + \Delta_2) + 2 + \frac{\Delta_1^2 + 2\Delta_1 + \Delta_2^2 + 1}{4}\left(\sigma^{-1}_{G_2}(y_1) + (s_2 - 1)\right) + \frac{\Delta_2^2 + 2\Delta_2 + \Delta_1^2 + 1}{4}\left(\sigma^{-1}_{G_1}(x_1) + (s_1 - 1)\right) \\
&\quad + \frac{\Delta_1^2 + \Delta_2^2}{4}\left((s_2 - 1)\sigma^{-1}_{G_1}(x_1) + (s_1 - 1)\sigma^{-1}_{G_2}(y_1) + (s_1 - 1)(s_2 - 1)\right) \\
&= RD^+_{HF}(G_1) + RD^+_{HF}(G_2) + \frac{s_2(\Delta_1^2 + \Delta_2^2) + 2(4\Delta_1 + \Delta_2) + 5}{4}\sigma^{-1}_{G_1}(x_1) + \frac{s_1(\Delta_1^2 + \Delta_2^2) + 2(\Delta_1 + 4\Delta_2) + 5}{5}\sigma^{-1}_{G_2}(y_1) + (\Delta_1^2 + \Delta_2^2) \\
&\quad + 2(\Delta_1 + \Delta_2) + 2 + (s_1 - 1)\frac{\Delta_2^2 + 2\Delta_2 + \Delta_1^2 + 1}{4}(s_2 - 1)\frac{\Delta_1^2 + 2\Delta_1 + \Delta_2^2 + 1}{4} + (s_1 - 1)(s_2 - 1)\frac{\Delta_1^2 + \Delta_2^2}{4}.
\end{aligned}
$$

□

3 Thorn graph

Remark:

Let T be a thorn graph of a (s, t)- graph G denote

$$l_1 = \sum_{i=1}^{s} r_i, \quad l_2 = \sum_{i=1}^{s} r_i^2, \quad l_3 = \sum_{i=1}^{s} r_i^3, \quad l_4 = \sum_{1\le i\le j\le s} r_i r_j, \quad l_5 = \sum_{1\le i\le j\le s} \frac{r_i + r_j}{d_G(x_j, x_j)}$$

$$l_6 = \sum_{1\le i\le j\le s} \frac{r_i r_j}{d_G(x_j, x_j)}, \quad l_7 = \frac{r_i^2 + r_j^2}{d_G(x_j, x_j)}, \quad l_8 = \frac{r_i r_j(r_i + r_j)}{d_G(x_j, x_j)}$$

Theorem 2. Let T be a thorn graph of G with $r_i, 1 \le i \le s$ pendent edges on the vertices $x_1, x_2, \cdots, x_s$ respectively. Then $RD^*_{HF}(T) \le \frac{1}{2} RD^+_{HF}(G) + \frac{(s+3)\Delta^2+s+2}{4} l_1 + \left(\frac{\Delta^2}{2} + \frac{6\Delta+1}{4}\right) l_2 + \frac{3l_3}{4} + \frac{l_4}{4} + \left(\frac{\Delta^2+8\Delta+1}{4}\right) l_5 \left(\frac{4\Delta+1}{2}\right) l_6 + l_7 + \frac{l_8}{4}$.

Proof. Let T be the thorn graph of G with $r_i, 1 \le i \le s$ parameters and let $\{x_1, x_2, \cdots, x_s\}$ be the vertices of G and $r_{ik}, 1 \le i < s$ be pendent vertices adjacent to x_i, where k is any non-negative integer.

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Volume 02 Issue 08 August-2026 | ISSN: 3108-1762 (Online) | Impact Factor: 3.8
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$$T_1 = \sum_{1\le i\le j\le s} \frac{(d_G(x_i) + r_i)^2 + (d_G(x_j) + r_j)^2}{d_G(x_i, x_j)}$$

$$= \sum_{1\le i\le j\le s} \frac{d_G^2(x_i) + d_G^2(x_j)}{d_G(x_i, x_j)} + \sum_{1\le i\le j\le s} \frac{r_i^2 + r_j^2}{d_G(x_i, x_j)} + \sum_{1\le i\le j\le s} \frac{2(r_i d(x_i) + r_j d(x_j))}{d_G(x_i, x_j)}$$

$$= RD^+_{HF}(G) + \sum_{1\le i\le j\le s} \frac{r_i^2 + r_j^2}{d_G(x_i, x_j)} + \sum_{1\le i\le j\le s} \frac{2(r_i d(x_i) + r_j d(x_j))}{d_G(x_i, x_j)}.$$

$$T_2 = \sum_{\substack{1\le i\le s\\ 1\le k\le r_i}} \frac{(d_G(x_i) + r_i)^2 + d_T^2(r_{ik})}{d_T(x_i, r_{ik})}$$

One can find that the distance from a vertex x_i to r_{ik} is 1 and degree of a vertex r_{ik} is T is 1. Thus

$$T_2 = \sum_{i=1}^{s} r_i(d_G^2(x_i) + r_i^2 + 2r_i d_G(x_i) + 1)$$

$$= \sum_{i=1}^{s} (r_i d_G^2(x_i) + 2r_i^2 d_G(x_i) + r_i^3 + r_i).$$

$$T_3 = \sum_{\substack{1\le i\le s\\ 1\le k<l\le r_i}} \frac{d_T^2(r_{ik}) + d_T^2(r_{il})}{d_T(r_{ik}, r_{il})}$$

$$= \frac{1}{2}\sum_{i=1}^{s} \binom{r_i}{2}$$

$$= \frac{1}{2}\sum_{i=1}^{s} \left(\frac{r_i(r_i - 1)}{2}\right)$$

$$= \frac{1}{4}\left(\sum_{i=1}^{s} r_i^2 - \sum_{i=1}^{s} r_i\right).$$

$$T_4 = \sum_{\substack{1\le i\le j\le s\\ 1\le k\le r_j}} \frac{(d_G(x_i) + r_i)^2 + d_T^2(r_{jk})}{d_T(x_i, r_{jk})}.$$

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In this case, $d_T(x_i, r_{jk}) = d_G(x_i + x_j) + 1$ and degree of r_{jk} is 1, Thus

$$T_4 = \sum_{1\le i\le j\le s} \frac{(d_G(x_i) + r_i)^2 + 1}{d_G(x_i, x_j) + 1} = \sum_{1\le i\le j\le s} \frac{d_G^2(x_i) + r_i^2 + 2r_i d_G(x_i) + 1}{d_G(x_i, x_j) + 1}$$

By Jenson's inequaltiy, we have

$$\frac{1}{d_G(x_i, x_j) + 1} \le \frac{1}{4d_G(x_i, x_j)} + \frac{1}{4} \text{ with equality } \Leftrightarrow d_G(x_i, x_j) = 1.$$

Thus

$$T_4 \le \sum_{1\le i\le j\le s} \frac{d_G^2(x_i) + r_i^2 + 2r_i d_G(x_i) + 1}{4d_G(x_i, x_j)} + \sum_{1\le i\le j\le s} \frac{d_G^2(x_i) + r_i^2 + 2r_i d_G(x_i) + 1}{4d_G(x_i, x_j)} = L_1 + L_2,$$

where

$$L_1 = \sum_{1\le i\le j\le s} \frac{r_j(d_G^2(x_i) + r_i^2 + 2r_i d_G(x_i) + 1) + r_i(d_G^2(x_j) + r_j^2 + 2r_j d_G(x_j) + 1)}{4d_G(x_i, x_j)}$$
$$= \sum_{1\le i\le j\le s} \frac{(r_j d_G^2(x_i) + r_i d_G^2(x_j) + r_j r_i^2 + r_i r_j^2 + 4r_i r_j(d_G(x_i) + d_G(x_j)) + r_i + r_j)}{4d_G(x_i, x_j)}.$$

hence

$$L_1 \le \sum_{1\le i\le j\le s} \frac{r_j\Delta^2 + r_i\Delta^2 + r_j r_i^2 + r_i r_j^2 + 8r_i r_j\Delta + r_i + r_j}{4d_G(x_i, x_j)} = \sum \frac{(\Delta^2 + 1)(r_i + r_j) + r_i r_j(r_i + r_j) + 8r_i r_j\Delta}{4d_G(x_i, x_j)}.$$

But

$$L_2 = \sum_{1\le i\le j\le s} \frac{d_G^2(x_i) + r_i^2 + 2r_i d_G(x_i) + 1}{4d_G(x_i, x_j)}$$
$$\le \frac{1}{4}\Big[\sum (\Delta^2 + r_i^2 + 2\Delta r_i + 1)\Big]$$
$$= \frac{1}{4}\sum \Big((\Delta^2 + 1) + r_i^2 + 2\Delta r_i\Big)$$
$$= \frac{1}{4}\Big[(\Delta^2 + 1)\Big(\sum_{i=1}^{s} r_i^2 - \sum_{i=1}^{s} r_i\Big) + \Big(\sum_{i=1}^{s} r_i\Big)^2 - \sum_{i=1}^{s} r_i^2 + 2\Delta\Big(\sum_{i=1}^{s} r_i\Big)^2 - 2\Delta\sum_{i=1}^{s} r_i^2\Big].$$

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Thus

$$T_4 = \sum_{1\le i<j\le s} \frac{(\Delta^2+1)(r_i+r_j) + r_ir_j(r_i+r_j) + 8r_ir_j\Delta}{4d_G(x_i,x_j)} + \left(\frac{\Delta^2+1}{4}\right)\left[s\sum_{i=1}^{s} r_i - \sum_{i=1}^{s} r_i\right] + \frac{1}{4}\left[\Big(\sum_{i=1}^{s} r_i\Big)^3 - \sum_{i=1}^{s} r_i^3\right] + \left(\frac{\Delta}{2}\right)\left[\Big(\sum_{i=1}^{s} r_i\Big)^2 - \sum_{i=1}^{s} r_i^2\right]$$

$$\begin{aligned} T_5 &= \sum_{\substack{1\le i<j\le s\\ 1\le k\le r_i\\ 1\le l\le r_j}} \frac{d_T^2(r_{ik}) + d_T^2(r_{jl})}{d_T(r_{ik}, r_{jl})} \\ &= \sum_{1\le i<j\le s} 2\frac{r_ir_j}{d_G(x_i,x_j)+2} \\ &\le \sum_{1\le i<j\le s} \frac{r_ir_j}{2d_G(x_i,x_j)} + \sum_{1\le i<j\le s} \frac{r_ir_j}{4} \end{aligned}$$

By the definition of RD^*_{HF} for the thorn graph T, we obtain

$$\begin{aligned} RD^*_{HF}(T) \le{}& \frac{1}{s}\Bigg[RD^+_{HF}(G) + \sum_{1\le i<j\le s}\frac{r_i^2+r_j^2}{d_G(x_i,x_j)} + \sum_{1\le i<j\le s}\frac{2\Delta(r_i+r_j)}{d_G(x_i,x_j)} \\ &+ \sum_{i=1}^{s}(\Delta^2 r_i + 2\Delta r_i^2 + r_i^3 + r_i) + \frac{1}{4}\sum_{i=1}^{s} r_i^2 - \frac{1}{4}\sum_{i=1}^{s} r_i \\ &+ \sum_{1\le i<j\le s}\frac{(\Delta^2+1)(r_i+r_j) + r_ir_j(r_i+r_j) + 8r_ir_j\Delta}{4d_G(x_i,x_j)} \\ &+ \left(\frac{\Delta^2+1}{4}\right)\left[s\sum_{i=1}^{s} r_i - \sum_{i=1}^{s} r_i\right] + \frac{1}{4}\left[\Big(\sum_{i=1}^{s} r_i\Big)^3 - \sum_{i=1}^{s} r_i^3\right] \\ &+ \left(\frac{\Delta}{2}\right)\left[\Big(\sum_{i=1}^{s} r_i\Big)^2 - \sum_{i=1}^{s} r_i^2\right] + \sum_{1\le i<j\le s}\frac{r_ir_j}{2d_G(x_i,x_j)} + \sum_{1\le i<j\le s}\frac{r_ir_j}{4}\Bigg] \\ ={}& \frac{1}{s}\Bigg[RD^+_{HF}(G) + l_7 + 2\Delta l_5 + \Delta^2 l_1 + 2\Delta l_2 + l_3 + l_1 + \frac{l_2}{4} - \frac{l_1}{4} \\ &+ \frac{(\Delta^2+1)}{4}l_5 + \frac{l_8}{4} + \frac{8\Delta}{4}l_6 + \frac{(\Delta^2+1)}{4}[sl_1 - l_2] + \frac{1}{4}[l_1^3 - l_3] \\ &+ \left(\frac{\Delta}{2}\right)[l_1^2 - l_2] + \frac{l_6}{2} + \frac{l_4}{4}\Bigg]. \end{aligned}$$

$$RD^*_{HF}(T) \le \frac{1}{s}\Bigg[RD^+_{HF}(G) + \left(\frac{(s+3)\Delta^2 + s + 2}{4}\right)l_1 + \frac{\Delta l_1^2}{2} + \frac{l_1^3}{4} + \left(\frac{6\Delta+1}{2}\right)l_2 + \frac{3l_3}{4} + \frac{l_4}{4} + \left(\frac{\Delta^2+8\Delta+1}{4}\right)l_5 + \left(\frac{4\Delta+1}{2}\right)l_6 + l_7 + \frac{l_8}{4}\Bigg].$$

□

Let T be a thorn graph of G with r parameters on every vertex of G. Then by remark.
$l_1 = sr, l_2 = sr^2, l_3 = sr^3, l_4 = (\frac{s}{2})r^2$

$$l_5 = 2r \sum_{1 \le i \le j \le s} \frac{1}{d_G(x_i, x_j)} = 2rH(G)$$

$$l_6 = r^2 \sum_{1 \le i \le j \le s} \frac{1}{d_G(x_i, x_j)} = r^2H(G)$$

$$l_7 = 2.r^2 \sum_{1 \le i \le j \le s} \frac{1}{d_G(x_i, x_j)} = 2r^2H(G)$$

$$l_8 = 2.r^3 \sum_{1 \le i \le j \le s} \frac{1}{d_G(x_i, x_j)} = 2r^3H(G)$$

Thus by Theorem 1 , we have

$$RD^*_{HF}(G) \le \frac{1}{2}\Big[RD^+_{HF}(G) + \frac{\big((s+3)\Delta^2 + s + 2\big)sr}{4} + \frac{sr^2(2\Delta s + 6\Delta + 1)}{4} + \frac{sr^3}{4}(s^2+3) + \frac{s(s-1)}{8}r^2 + \frac{rH(G)}{2}\big(\Delta^2 + 4\Delta(r+2) + r(r+2) + 1\big)\Big].$$

4 Conclusion

In this paper, upper bounds for the product Harary–Furtula index $RD*_{HF}(G)$ were established for splice and thorn graphs. The degree and distance contributions arising from these graph constructions were analyzed to obtain the required bounds. The results provide useful relationships between the product Harary–Furtula index and the structural parameters of the underlying graphs.

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